

THE AUBRY-MATHER THEOREM FOR DRIVEN GENERALIZED ELASTIC CHAINS

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ABSTRACT. We consider uniformly (DC) or periodically (AC) driven generalized infinite elastic chains (a generalized Frenkel-Kontorova model) with gradient dynamics. We first show that the union of supports of all the invariant measures, denoted by \mathcal{A} , projects injectively to a dynamical system on a 2-dimensional cylinder. We also prove existence of ergodic invariant measures supported on a set of rotationally ordered configurations with an arbitrary (rational or irrational) rotation number. This shows that the Aubry-Mather structure of ground states persists if an arbitrary AC or DC force is applied. The set \mathcal{A} attracts almost surely (in probability) configurations with bounded spacing. In the DC case, \mathcal{A} consists entirely of equilibria and uniformly sliding solutions. The key tool is a new weak Lyapunov function on the space of translationally invariant probability measures on the state space, which counts intersections.

1. INTRODUCTION

We consider the equation

$$(1.1) \quad \frac{d}{dt}u_j(t) = -V_2(u_{j-1}(t), u_j(t)) - V_1(u_j(t), u_{j+1}(t)) + F(t), \quad j \in \mathbb{Z}$$

such that $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 , periodic ($V(u+1, v+1) = V(u, v)$), and satisfies the twist condition $V_{12}(u, v) \leq -\delta < 0$ for some fixed $\delta \in \mathbb{R}$.

We assume that F is either constant (the DC case) or 1-periodic in t (the AC case). In the AC case, we further assume that V, F are real analytic.

1.1. The background and motivation. The case $F = 0$ is the standard setting of the well-known Aubry-Mather theory ([4], [16], Section 9.3), as the solutions of $du/dt = 0$ correspond to orbits of an area-preserving twist map f_V on a cylinder whose generating function is V . The Aubry-Mather theory describes minimizing configurations/orbits and measures, and so simultaneously characterizes physically important "ground states" of (1.1) and describes "KAM-circles" and what is left of them for f_V .

Our goal is to contribute towards the description of the dynamics of (1.1) on the entire state-space, allowing a fairly general family of initial conditions. We are, however, inspired by results on asymptotics of one-dimensional reaction-diffusion equations ([9], [15]), their braid dynamics ([13]), stability of synchronization phenomena ([10]); in the Hamiltonian dynamics setting the Mather's measure-theoretical approach to minimizing orbits and its extensions ([19]); PDE extensions of Aubry-Mather theory ([18]); results on spatio-temporal entropy of extended PDE's ([20],

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[29]); all related to what is actually known on the behavior of (1.1) either numerically, physically ([7]) or analytically ([4], [5], [14], [22], [23]). Before stating precisely the actual results, we will try to put them in a wider context of these topics.

In the "standard" case $V(u, v) = (u - v)^2/2 - W(u)$, where $W(u + 1) = W(u)$ and $W : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 , the equation (1.1) is a spatial discretization of the scalar reaction-diffusion equation

$$u_t = u_{xx} + W'(u),$$

and it is closely related to a more general reaction-diffusion equation

$$(1.2) \quad u_t = u_{xx} + f(x, u, u_x),$$

where f is 1-periodic in the first coordinate. Both (1.1) and (1.2) are scalar and order-preserving (or monotone, or cooperative, see [27]). The well-known Fiedler and Mallet-Paret Poincaré-Bendixson theorem [9] showed that the asymptotics of (1.2) on any bounded domain is 2-dimensional, which is a basis of a relatively complete understanding of dynamics of (1.2) on bounded domains ([15] and references therein). The key tool they applied is the intersection-counting function $(u, v) \rightarrow z(u - v)$, which is a discrete Lyapunov function. That means that $t \mapsto z(u(t) - v(t))$ is non-increasing along two solutions of (1.2). Furthermore, if $u(t_0) - v(t_0)$ has a singular zero (we also say that $u(t_0)$ and $v(t_0)$ intersect non-transversally), $z(u(t) - v(t))$ strictly drops at $t = t_0$.

These ideas have not in our knowledge been applied to the equation (1.2) on the entire \mathbb{R} , considered as an "extended system" (i.e. without assuming $u(x) \rightarrow 0$ as $x \rightarrow \infty$). The reason is that the intersection-counting function z is typically not finite when u, v are defined on the entire \mathbb{R} , thus all bounded-domain tools fail. We believe that the right approach in the extended-domain case is to consider the extension of the intersection-counting function z to the space of translationally invariant measures on the state space.

In this paper we pursue this idea for a technically simpler discrete-space system (1.1), and consider a function

$$(\mu^1, \mu^2) \mapsto Z(\mu^1, \mu^2)$$

which counts the average number of intersections of configurations with respect to two translationally invariant probability measures on the space of configurations $(u_i)_{i \in \mathbb{Z}}$. We will show that $t \mapsto Z(\mu^1(t), \mu^2(t))$ is a weak Lyapunov function (for details see Section 5), where $\mu^1(t), \mu^2(t)$ are evolutions of measures with respect to (1.1). We believe these ideas can also be applied also to (1.2) and give fuller insight on the dynamics of (1.2) on unbounded domain. In particular, we conjecture that all our results (with appropriate rephrasing) hold also for (1.2).

Van der Berg, Ghrist, Vandervorst and Wójcik studied in several papers topological "braid" properties of intersections of solutions of (1.2) on bounded domains and its spatial discretization ([13] and references therein). Our approach may lead to extension of their ideas to unbounded domains.

It has been noted that synchronized solutions of the extended reaction-diffusion equation in more spatial dimensions (≥ 2) are often stable and locally attracting (see e.g. [10] for recent results and references). We argue that it is possible to study this phenomenon also in the spatial dimension 1. As compared to [11], we also allow unbounded solutions $u_i(t)$ (by identifying u and $u + 1$ and considering the dynamics in an appropriate quotient space), and also study non-autonomous systems (AC forcing). We eventually reduce our dynamics to a 2d flow (DC case) or map (AC case) with non-trivial dynamics (the spatially 1d dynamics in [10] reduces to a trivial 1d flow). A heuristic explanation for stability of 1d synchronization is now as follows: the synchronized solutions are the solutions in the support of invariant measures μ such that the self-intersecting function $Z(\mu, \mu)$ is equal to zero (we

consider measures invariant both for spatial translations and time evolution). As the function $t \mapsto Z(\mu(t), \mu(t))$ is non-decreasing, such solutions are expected to be at least Lyapunov stable. We develop a rigorous approach to this idea.

The Aubry-Mather theory of ground states of (1.1) in the case $F = 0$ has been extended to more dimensional lattices and PDEs ([18] and references therein). In our knowledge, generalizations typically deal with existence of ground states with an arbitrary generalized rotation number (rational or irrational). In the PDE setting, this leads to construction of various solutions of stationary PDEs. In our view, it is somewhat unexpected that the Aubry-Mather stationary structure seems to be preserved in a not entirely obvious way for an evolution equation (1.1) with arbitrarily strong, and even time-periodic forcing.

Finally, let us come back to the actual dynamics of (1.1). The Frenkel-Kontorova model is one of the most studied physical models, as it reasonably realistically models a variety of physical phenomena, and is perhaps a simplest model with sufficiently rich structure allowing lots of "metastable states", various phase transitions such as "pinning-depinning", etc. ([7] and references therein). It has been long noted in the physics literature that the dynamics of (1.1) seems to be attracted to low-dimensional invariant manifolds, or to spatially synchronize. Baesens and MacKay proved that in the DC case spatially periodic initial conditions converge either to equilibria or to uniformly sliding solutions ([4], [5]; see below for details). Hu, Bambi and Qin extended some of those results to show existence and properties of spatially periodic, ordered solutions in the AC case ([14]), and Qin studied existence and properties of uniformly sliding solutions with both rational and irrational rotation number in the DC case ([22], [23]).

Here we attempt to give a broader perspective on the dynamics of (1.1) on a large state-space, and generalize most of the mentioned results. Firstly, we consider dynamics on the space of configurations of bounded spacing (configurations $(u_i)_{i \in \mathbb{Z}}$ such that $\sup_{i \in \mathbb{Z}} |u_{i+1} - u_i| < \infty$), which is more general than the configurations of bounded width (i.e. such that $\sup_{i \in \mathbb{Z}} |u_i - i\rho| < \infty$ for some $\rho \in \mathbb{R}$) rigorously studied so far. This is important, as it includes configurations connecting two configurations with different rotation number, and various other relevant situations. We then show existence of a 2d invariant set of configurations in both AC and DC case which in a physically meaningful way attracts configurations with bounded spacing. Finally, we show existence of ordered configurations with an arbitrary (rational or irrational) rotation number.

1.2. The statement of results. The state space for the dynamics of (1.1) of configurations of bounded spacing will be denoted by X , and defined precisely as follows:

$$\begin{aligned} K_n &= \{u \in \mathbb{R}^{\mathbb{Z}}, \text{ such that } \sup_{i \in \mathbb{Z}} |u_{i+1} - u_i| \leq n\}, \\ X &= \bigcup_{n=1}^{\infty} K_n, \end{aligned}$$

with the induced product topology. We will use the quotient sets $\mathcal{X} = X/R$, $\mathcal{K}_n = K_n/R$, where $Ru = u + 1$ is a translation commuting with (1.1) (for details see Section 5).

The system (1.1) generates a continuous global semiflow and a local flow on \mathcal{X} (see Section 4), which we denote by φ (or φ^t for a given t). Of key importance will be the translation $T : X \rightarrow X$ defined naturally as $(Tu)_i = u_{i-1}$, commuting with (1.1) and φ .

The Borel probability measures on \mathcal{X} will be called *measures with bounded spacing*. We will in particular study measures with bounded spacing, invariant with

respect to both the translation T and the semiflow φ^t , called (φ, T) -invariant. In the DC case, the measures will be invariant with respect to φ^t for all $t \in \mathbb{R}$, and in the AC case with respect to $t \in \mathbb{Z}$. Denote the sets of such measures on $\mathcal{K}_n, \mathcal{X}$ by $\mathcal{E}(\mathcal{K}_n), \mathcal{E}(\mathcal{X})$. Let

$$(1.3) \quad \mathcal{A} = \bigcup_{\mu \in \mathcal{E}(\mathcal{X})} \text{supp}(\mu),$$

where $\text{supp}(\mu)$ denotes the support of μ . Our first main result is the following:

Theorem 1. *There is a continuous injective projection $\pi : \mathcal{A} \rightarrow \mathbb{T}^1 \times \mathbb{R}$.*

The injective projection will be given by

$$(1.4) \quad \pi(u) = (u_0, u_1 - u_0).$$

As the projection π is injective, we can pull the translation T , and the semiflow φ^t in the DC case, or the map φ^1 to the 2-dimensional set $\pi(\mathcal{X})$. We introduce the following notation:

$$(1.5) \quad \begin{aligned} \tilde{\mathcal{X}} &= \pi(\mathcal{X}), \\ \tilde{\mathcal{K}}_n &= \pi(\mathcal{K}_n), \\ \tilde{T} &= \pi \circ T \circ \pi^{-1}, \\ \tilde{\varphi}^t &= \pi \circ \varphi^t \circ \pi^{-1}. \end{aligned}$$

Theorem 2. *For each $n \in \mathbb{N}$, the map \tilde{T} is a homeomorphism of $\tilde{\mathcal{K}}_n$, in the DC case $\tilde{\varphi}^t$ is a continuous flow on $\tilde{\mathcal{K}}_n$, in the AC case $\tilde{\varphi}^1$ is a homeomorphism on $\tilde{\mathcal{K}}_n$, and \tilde{T} and $\tilde{\varphi}^t$ (respectively $\tilde{\varphi}^1$) commute.*

We propose to call the pair $(\tilde{T}, \tilde{\varphi})$ in the DC case the *characteristic flow-map*, and the pair $(\tilde{T}, \tilde{\varphi}^1)$ in the AC case the *characteristic maps*.

We give a detailed example in Section 2 for the case $F = 0$, and relate our main results to the Aubry-Mather theory. In that case in particular, $\tilde{\mathcal{X}}$ is the entire cylinder $\mathbb{T}^1 \times \mathbb{R}$ if V is close to integrable, $\tilde{\varphi}^t$ is the identity, and \tilde{T} is diffeomorphic to an area-preserving twist map whose generating function is V ([16]).

We do not know whether in general the maps \tilde{T} and $\tilde{\varphi}^t$ can be naturally extended to the entire cylinder $\mathbb{T}^1 \times \mathbb{R}$. Also unlike in the equilibrium case $F = 0$, we do not a-priori know $\tilde{T}, \tilde{\varphi}^t$. We give some conjectures in Section 6. We can, however, show that $\tilde{\mathcal{X}} \subset \mathbb{T}^1 \times \mathbb{R}$ is a "rich" set for all F , as it contains analogues to Aubry-Mather sets of minimizing configurations/orbits with an arbitrary *rotation number* (called also *mean spacing*) found in the case $F = 0$, and to Mather's minimizing measures ([19]). (We postpone definitions of rotation number, a (φ, T) -ergodic measure, a rotationally ordered configuration, and totally ordered orbit to Sections 5, 6.)

Theorem 3. *Given any $\rho \in \mathbb{R}$, there exists a (φ, T) -ergodic measure on \mathcal{X} supported on a set of rotationally ordered configurations with the rotation number ρ .*

In the DC case, we can completely describe the set of all (φ, T) -invariant and ergodic measures:

Theorem 4. *Assume F is DC. Then every (φ, T) -ergodic measure on \mathcal{X} is supported either on a set of equilibria, or on a single totally ordered periodic orbit.*

The totally ordered periodic orbits in Theorem 4 are in the literature known as uniformly sliding states ([4], [5], [22], [23]). We show in Section 9 that the known results on uniformly sliding states follow as a direct corollary from our considerations.

Finally, we address asymptotics of the dynamics (1.1) relative to the set \mathcal{A} defined by (1.3).

1.3. The spatio-temporal attractor. We propose a notion of attractiveness which is weaker than the usual notions of attractor or union of ω -limit sets, but for extended systems such as (1.1) often in practice (physical systems or numerical simulations) indistinguishable from them. Imagine a physical situation in which we observe dynamics of (1.1) on \mathbb{Z} (or a very large subset of \mathbb{Z}), and that no position in space is preferred. Consider also that an initial condition $u(0)$ emerges as a realization of some spatially homogenous random process. For example, we can choose $u(0)$ as a double-infinite sequence so that u_i (or $u_{i+1} - u_i$) is chosen randomly and independently in each step $i \in \mathbb{Z}$ from the same bounded measurable subset of \mathbb{R} . We can study that as analyzing the dynamics (1.1) with respect to μ -a.e initial condition, where μ is a T -invariant probability measure on \mathcal{X} . We propose a notion of spatio-temporal attractiveness, and say that a set \mathcal{A} is a spatio-temporal attractor, if given any T -invariant probability measure μ , then for μ -a.e. initial condition u and with density of times t equal to 1, $\varphi^t(u)$ is arbitrarily close to \mathcal{A} . More precisely, given any $S > 0$, we define the Borel probability measure $P_{\mu,S}$ on $\mathcal{X} \times [0, S]$ as the product measure $\mu \times \lambda_S$, where $\lambda_S = (1/S)\lambda|_{[0,S]}$ is the normed Lebesgue measure λ restricted to $[0, S]$, and define the attractor as follows:

Definition 1. A set \mathcal{A} is a spatio-temporal attractor for (1.1) on \mathcal{X} , if given any T -invariant probability measure μ on \mathcal{X} and any open neighbourhood U of \mathcal{A} ,

$$(1.6) \quad \lim_{S \rightarrow \infty} P_{\mu,S}(\{(u, t), \varphi^t(u) \in U\}) = 1.$$

We finally in Section 10 we show that the set \mathcal{A} described in Theorems 1-4, is indeed a spatio-temporal attractor.

Theorem 5. The set \mathcal{A} defined with (1.3) is the spatio-temporal attractor for (1.1) on \mathcal{X} .

Lastly, we comment on the spatio-temporal entropy of (1.1). A corollary of Theorem 4 is that the spatio-temporal entropy of (1.1) on \mathcal{X} in the DC case is zero (see e.g. [20], [29] for definitions). A proof follows from the variational principle for metric and topological spatio-temporal entropy and our description of all spatio-temporal invariant measures; we intend to report details separately. This complements well the existing results for PDE systems such as (1.2). Specifically, Zelik [29] showed that extended gradient systems (the case $F = 0$ of (1.1)) have spatio-temporal entropy zero, Mielke and Zelik [20] constructed 1d non-autonomous PDE examples with positive entropy, and Turaev and Zelik [28] proved positive entropy for a 1d complex system.

1.4. The structure of the paper. In Section 2 we as an example summarize the known facts on the dynamics (1.1) in the "extended gradient" case $F = 0$, and relate them to our results. In Sections 3, 4 we develop tools on the dynamics of the zero-set of a non-autonomous system of infinitely many linear ODEs, and as a corollary on dynamics of intersections of a cooperative system of infinitely many ODEs. Here we extend some of the known results ([5], [26], [27]) to systems of infinitely many ODEs, less smooth than e.g. in [26], and on a general space of solutions (the Fréchet space of configurations of sub-exponential growth at infinity). As such, our results could be of independent interest. We postpone most of the technicalities to Appendices 1,2, dealing in particular with some exceptional, "degenerate" situations due to the infinite number of equations. In Section 5 we introduce the semiflow on the space of translationally invariant measures on \mathcal{X} , and show that the average-number-of-intersections-counting function $Z(\mu_1, \mu_2)$ is non-increasing and strictly decreasing if u, v in the supports of μ_1, μ_2 have a non-transversal intersection (or, using the alternative terminology of the paper, $u - v$ has a singular zero). Once all the tools

are in place, the proofs of Theorems 1–5 are relatively straightforward and given in Sections 6–10, including a more detailed discussion of uniformly sliding states.

2. EXAMPLE: DYNAMICS WITHOUT DRIVING AND AUBRY-MATHER THEORY

In this section we relate our results to the known results in the case of no driving $F = 0$. Our assumptions on the function V are the standard setting of the Aubry-Mather theory. Let us first consider the equilibria $du/dt = 0$ of (1.1), denoted by $\mathcal{E} \subset \mathbb{R}^{\mathbb{Z}}/R$, where we again identify u and $Ru = u + 1$ and consider the quotient set. It is straightforward to deduce from the conditions on V that the projection $\pi : \mathcal{E} \rightarrow \mathbb{T}^1 \times \mathbb{R}$ is a homeomorphism (again considering the induced product topology on \mathcal{E}). Also the map $\tilde{T} : \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{T}^1 \times \mathbb{R}$ is C^2 , diffeomorphic to the twist map generated by V ([16], Section 9.3). The diffeomorphism is given by the discrete Legendre transform, which is for example identity in the "standard" case

$$(2.1) \quad V(u, v) = (v - u)^2/2 + W(u),$$

where $W(u + 1) = W(u)$ and W is a C^2 function.

It is however not a-priori clear that all the (φ, T) -invariant measures of (1.1) with $F = 0$ are supported on \mathcal{E} . The system (1.1) is an example of an extended gradient system, i.e. a system which has a formal Lyapunov function (in our case the formal sum $\sum_{i=-\infty}^{\infty} V(u_i, u_{i+1})$), and is a gradient system when reduced to a finite domain. It has been shown that extended gradient systems can have more complex dynamics than what is by LaSalle principle possible for gradient systems ([11], [12], [25]), including possible existence of non-stationary orbits in ω -limit sets.

In [25] we however showed the following:

Theorem 6. *All (φ, T) -invariant measures of (1.1) on \mathcal{X} in the case $F = 0$ are supported on \mathcal{E} .*

The key tool is an observation that the function $\mu \mapsto \int V(u_0, u_1) d\mu$ is a Lyapunov function on the space of T -invariant probability measures \mathcal{X} , so LaSalle principle must hold for the evolution of T -invariant measures.

We conclude that Theorems 1-3 do hold in the case $F = 0$, where the measures in Theorem 3 are for example Mather's minimizing measures ([6], [19]).

Let us briefly discuss the question whether and when $\pi : \mathcal{A} \rightarrow \mathbb{T}^1 \times \mathbb{R}$ is a homeomorphism, where \mathcal{A} is the support of all (φ, T) -invariant measures of bounded spacing as defined in (1.3). One can see that π is a homeomorphism if and only if $\mathcal{E} \subset \mathcal{X}$. This is true because \tilde{T} is diffeomorphic to an area-preserving map, so we can pull back a Lebesgue measure and construct (φ, T) -invariant measures supported on any closed subset of $\mathcal{E} \cap \mathcal{X}$. In general, $\mathcal{E} \subset \mathcal{X}$ is true if and only if the area-preserving twist map f_V generated by V is close enough to integrable. For example, in the standard case (2.1), $\mathcal{E} \subset \mathcal{X}$ holds if and only if f_V has homotopically non-trivial invariant circles, otherwise one can construct "accelerating" invariant measures of f_V ; i.e. there are T -invariant measures in \mathcal{E} which do not have bounded spacing ([24]).

Finally, let us consider asymptotics. The following slightly stronger version of Theorem 5 in the case $F = 0$ was shown in [25] and extended in [11], [12] for much more general systems:

Theorem 7. *Consider the dynamics (1.1) with $F = 0$ on \mathcal{X} .*

(i) *If μ is a T -invariant Borel probability measure on \mathcal{X} , then the ω -limit set for μ -a.e. $u \in \mathcal{X}$ consists of equilibria.*

(ii) *Let $u \in \mathcal{X}$ and let U be any open neighborhood of \mathcal{E} . Then the set of times $t \in [0, \infty)$ such that $u(t) \in U$ has Banach density 1.*

We can rephrase Theorem 7, and say that the ω -limit set consists of equilibria either almost surely with respect to space, or with respect to time. One can not omit the "almost surely" part of the statement and still obtain a $2d$ union of limit sets either in Theorem 5 or 7, as it is most likely infinite-dimensional, analogously to the continuous space case ([21]).

3. THE DYNAMICS OF A ZERO-SET

Here we first consider in general terms properties of the zero-set of a solution of a linear, cooperative non-autonomous system of infinitely many ordinary differential equations. We show that the number of zeroes in a bounded domain can not increase and is strictly decreasing at singular zeroes. We then extend these ideas to systems of infinitely many equations, and establish a "zero-balance equation". Although some of the ideas presented here are essentially for finite dimensional systems already contained in [26], [27], our presentation follows more closely Angenent, Fiedler and Mallet-Paret [1], [2], [9] who developed it in the PDE setting. In particular, we use the PDE-terms "regular" and "singular zeroes", rather than "transversal" and "non-transversal intersections" of $v = 0$, as we will in a way consider the dynamics of the zero-set.

Let us consider a system of equations

$$(3.1) \quad \frac{d}{dt}w_j(t) = a_j(t)w_{j-1} + b_j(t)w_{j+1} + c_j(t)w_j, \quad j \in \mathbb{Z}$$

We assume that for some $T_0 < T_1$, the following holds:

$$(3.2) \quad \begin{aligned} & a_j, b_j, c_j \text{ continuous, uniformly bounded in } j, t \in [T_0, T_1], \\ & a_j, b_j \geq \delta > 0 \text{ for all } j, t \in [T_0, T_1]. \end{aligned}$$

The condition $a_j, b_j \geq \delta > 0$ means that the system (3.1) is cooperative ([26], [27]). We can consider (3.1) to be a discrete analogue of parabolic differential equations considered by Angenent (in real analytic case in [1], in C^2 case in [2]). We consider (3.1) on the space Y of configurations of sub-exponential growth. We define it as the set $Y \subset \mathbb{R}^{\mathbb{Z}}$ of $w = (w_i)_{i \in \mathbb{Z}}$ such that

$$\limsup_{j \rightarrow \infty} \frac{\log |w_j|}{|j|} = 0.$$

Indeed, Y is a Fréchet space generated by the family of norms

$$(3.3) \quad \|u\|_{n,\infty} = \sup_{j \in \mathbb{Z}} \exp(-|j|/n) |u_j|,$$

n a positive integer.

We first give definitions. We say that $(j, t) \in \mathbb{Z} \times [T_0, T_1]$ is a zero of a $w : [T_0, T_1] \rightarrow Y$, if the graph of $i \mapsto w_i(t)$ intersects $y = 0$ in the interval $[j, j+1]$ (i.e.

$$w_j(t) + (w_{j+1}(t) - w_j(t))x = 0$$

has a solution for $x \in [0, 1]$). We say that (j, t) is a singular zero, if $w_j(t) = 0$, and either of the following three cases holds:

$$\begin{aligned} w_{j+1}(t) &= 0, \text{ or} \\ w_{j-1}(t) &= 0, \text{ or} \\ w_{j+1}(t)w_{j-1}(t) &> 0 \end{aligned}$$

(these are the discrete space analogues of $w_x(t) = 0$); otherwise (j, t) is a regular zero. We denote by $Z, S, R \subset \mathbb{Z} \times [T_0, T_1]$ the sets of zeroes, singular zeroes and regular zeroes of a given $w(t)$. A singular zero has a degree k , if k is the maximal number of consecutive j such that (j, t) is a zero. Assume now $(j+1, t), \dots, (j+k, t) \in$

S is a regular zero of a finite degree k , $(j, t), (j + k + 1, t) \notin Z$. We distinguish two types of singular zeroes:

Type I, if $w_j(t)w_{j+k+1}(t) < 0$,

Type II, if $w_j(t)w_{j+k+1}(t) > 0$.

Note that for singular zeroes of Type I, the degree is $k \geq 2$, and for Type II, $k \geq 1$ (see Figure 1 in the Appendix 1). A zero can also be of degree ∞ , that is $w(t) \in Y$ such that for all $i \geq i_0$, $w_i(t) = 0$, or such that for all $i \leq i_0$, $w_i(t) = 0$, but w not identically 0. When discussing zeroes, we will need to consider this, intuitively degenerate and quite unlikely case separately with some care.

Given $w \in Y$, we define $z_j(w)$ to be 1 if w has a zero at j , otherwise 0. Let $z_{m,n} : Y \rightarrow \mathbb{N} \cup \{0\}$ be the zero-counting function

$$z_{m,n}(w) = \sum_{j=m}^{n-1} z_j(w),$$

where $m < n$ are integers.

We say that a solution of (3.1) is C^1 if it is C^1 in each coordinate. We can now state the discrete analogue of Angenent [2], Theorem D (i.e. for free boundary conditions not intersecting zero):

Lemma 1. *Assume $w : [T_0, T_1] \rightarrow Y$ is a C^1 solution of (3.1) satisfying (3.2), $m < n$ integers. Assume also $w_j(t) \neq 0$ for $(j, t) \in \{m, n\} \times [T_0, T_1]$. Then*

(i) *The number of zeroes $z_{m,n}(w(t))$ is non-increasing.*

(ii) *If $w(t_0)$ has a singular zero between m and n , then $z_{m,n}(w(t))$ is strictly decreasing at $t = t_0$.*

(iii) *If $w(t_0)$ has a singular zero between m and n , then there exists $\delta_0 > 0$ so that for all $0 < |\delta| \leq \delta_0$, $w(t_0 + \delta)$ has no singular zeroes between m and n .*

We postpone the proof (as well as all the other proofs in this section) to Appendix 1 as it is rather technical, and the approach is not related to the rest of the paper.

We would now like to make general statements on values of $z_{m,n}$ for arbitrary time and space intervals, including those when a zero is "crossing" a boundary $\{m, n\}$ of a segment, and the assumptions of Theorem 1 do not hold.

We will associate to a solution $w : [T_0, T_1] \rightarrow Y$ of (3.1) a family of functions $c_i(w; s, t)$, $i \in \mathbb{Z}$, $0 \leq s \leq t$, counting the number of times a zero enters minus the number of times it leaves the segment $i \geq 0$ during the time interval $(s, t]$ (a precise definition will be given below). In addition, we define $d_i(w; s, t)$, $i \in \mathbb{Z}$, $0 \leq s \leq t$ as the number of zeroes "disappearing" at the position i in the time interval $(s, t]$. We will omit the argument w when it is clear from the context.

As the definitions of c_i, d_i are intuitively straightforward but technically somewhat subtle, we give it axiomatically, and then prove their existence in Appendix 1.

(A1): The zero-balance equation: for all integers $m < n$ and all $T_0 \leq s < t \leq T_1$ the following holds:

$$(3.4) \quad z_{m,n}(w(t)) - z_{m,n}(w(s)) = c_m(s, t) - c_n(s, t) - \sum_{j=m}^{n-1} d_j(s, t).$$

(A2): Translation invariance: c_j, d_j are invariant for the translation T , that is

$$\begin{aligned} c_j(Tw; s, t) &= c_{j+1}(w; s, t), \\ d_j(Tw; s, t) &= d_{j+1}(w; s, t). \end{aligned}$$

- (A3): **Regularity of c :** If $w_j(t) \neq 0$ for each $t \in [t_0, t_1]$, then $c_j(t_0, t_1) = 0$.
 (A4): **Cardinality of \dot{d} :** If $\{(m, t_0), \dots, (m+k-1, t_0)\}$ is a zero of degree k , then there is $\delta_0 > 0$ so that for all $0 < \delta \leq \delta_0$,

$$\sum_{j=m}^{m+k-1} d_j(t_0 - \delta, t_0 + \delta) = \begin{cases} k-1 & \text{zero of Type I,} \\ k & \text{zero of Type II.} \end{cases}$$

If there are no singular zeros at j for $t \in [t_0, t_1]$, then $d_j(t_0, t_1) = 0$.

We again postpone the proof of the following to the Appendix:

Lemma 2. *We can associate to each C^1 solution $w : [T_0, T_1] \rightarrow Y$ of (3.1) satisfying (3.2) a family of functions $c_j(w; s, t) \in \mathbb{Z}$ and $d_j(w; s, t) \in \{0, 1, \dots\}$ satisfying (A1)-(A4).*

Now recall the partial order on $\mathbb{R}^{\mathbb{Z}}$:

$$w \geq v \text{ if for all } j, w_j \geq v_j.$$

The following well-known monotonicity property ([5], [27]) can be understood as a special case of the discussion above; for the sake of completeness a short proof is also in Appendix 1.

Lemma 3. *Assume $w : [T_0, T_1] \rightarrow Y$ is a solution of (3.1) satisfying (3.2). Then if $w(0) \geq 0$, then for all $t > 0$, $w(t) \geq 0$.*

4. THE DYNAMICS OF INTERSECTIONS

In this section we consider the dynamics of intersections $w = u^1 - u^2$ of two solutions u^1, u^2 of (1.1) and of derivatives $w = du/dt$ of a solution u of (1.1). We first show that the results of the previous section apply to $u^1 - u^2$ in both the AC and DC case, and to du/dt in the DC case. We then establish local and global existence of a continuous semiflow generated by (1.1). Finally in a series of four lemmas, we establish the key tool: that the singular zeroes of $u^1 - u^2$ and du/dt are persist for small perturbations in the product topology. Here we treat the cases of singular zeroes of finite and infinite degree separately, as the later case is technically more different.

4.1. Existence of semiflow. Recall the definition of K_n as the set of all $u \in \mathbb{R}^{\mathbb{Z}}$ such that $|u_{j+1} - u_j| \leq n$, and fix $n \in \mathbb{N}$.

Proposition 1. *Assume $u^1, u^2, u : [T_0, T_1] \rightarrow K_n$ are solutions of (1.1). Then $w = u^2 - u^1$ in all cases, and $w = du/dt$ in the DC case, are solutions of (3.1) with some a_i, b_i, c_i satisfying (3.2).*

Proof. If $w = u^2 - u^1$, we define

$$\begin{aligned} a_i &= \frac{V_2(u_{i-1}^2, u_i^1) - V_2(u_{i-1}^1, u_i^1)}{u_{i-1}^1 - u_{i-1}^2}, \\ b_i &= \frac{V_1(u_i^1, u_{i+1}^2) - V_1(u_i^1, u_{i+1}^1)}{u_{i+1}^1 - u_{i+1}^2}, \\ c_i &= \frac{V_2(u_{i-1}^2, u_i^2) - V_2(u_{i-1}^2, u_i^1) + V_1(u_i^2, u_{i+1}^2) - V_1(u_i^1, u_{i+1}^2)}{u_i^1 - u_i^2} \end{aligned}$$

(naturally extended to $a_i = -V_{12}$, $b_i = -V_{12}$, $c_i = -V_{11} - V_{22}$ in the cases $u_{i-1}^1 = u_{i-1}^2$, $u_{i+1}^1 = u_{i+1}^2$, $u_i^1 = u_i^2$ respectively). Then by the twist condition and the mean value theorem, $a_i, b_i \geq \delta > 0$, and by periodicity of V and a simple compactness argument, $a_i(t), b_i(t), c_i(t)$ are bounded, thus (3.2) holds. As $u^1(t), u^2(t) \in K_n$, the difference $j \mapsto |u_j^1(t) - u_j^2(t)|$ grows at most linearly in $|j|$, thus $u^1(t) - u^2(t) \in Y$.

Now let $w = du/dt$ in the DC case. Then by differentiating (1.1) we see that (3.1), (3.2) hold with

$$\begin{aligned} a_i &= -V_{12}(u_{i-1}, u_i), \\ b_i &= -V_{12}(u_i, u_{i+1}), \\ c_i &= -V_{22}(u_{i-1}, u_i) - V_{11}(u_i, u_{i+1}). \end{aligned}$$

□

We now establish that the equation (1.1) generates a smooth semiflow on $\mathcal{K}_n = K_n/R$, denoted by φ^t and φ .

Lemma 4. *The equation (1.1) generates a continuous semiflow on \mathcal{K}_n , and a continuous local flow on \mathcal{X} .*

For each $i \in \mathbb{Z}$, the solution $t \mapsto u_i(t)$ is C^2 ; and if F, V are real analytic, $t \mapsto u_i(t)$ is also real analytic, all on all open sets of all t for which the solution $u(t) \in \mathcal{X}$.

Proof. It is easy to show by applying the well-known results on existence of solutions on Banach spaces, that for $u(0) \in \mathcal{K}_n$ the solution of (1.1) exists locally, and $t \mapsto u(t) - u(0)$ is continuous in $l_\infty(\mathbb{Z})$. Thus (1.1) generates a local flow on \mathcal{X} .

We can show that the solution depends continuously on initial conditions in the product topology, by for example establishing local existence of solutions in any of the norms $\|\cdot\|_{n,\infty}$ introduced in the previous section (note that these norms induce the product topology on \mathcal{K}_n and \mathcal{X}). Finally, given a positive integer n , the property $u \in \mathcal{K}_n$ can be written as

$$Tu + n \geq u \geq Tu - n,$$

which is by Lemma 3 and Proposition 1 applied to $w(t) = (Tu + n)(t) - u(t)$, $w(t) = u(t) - (Tu - n)(t)$ invariant forward in time. We conclude that if $u(0) \in \mathcal{K}_n$, then $u(t) \in \mathcal{K}_n$ for all $t \geq 0$.

Regularity follows from (1.1). In the real analytic case we use real analyticity of F, V and check inductively that the derivatives satisfy the characterization of real analyticity ([17], Proposition 1.2.12). □

We will often fix an integer n and write \mathcal{K} omitting its superscript.

Generally the results to follow regarding the difference $u^1 - u^2$ will hold in both AC and DC cases, and the results regarding the derivative du/dt in the DC case only.

4.2. The AC and DC cases. We first show that singular zeroes of a finite degree persist for small perturbations in the product topology. Without loss of generality we consider zeroes at $t_0 = 0$.

Lemma 5. *Assume $u^1(0), u^2(0) \in \mathcal{K}$ such that $u^1(0) - u^2(0)$ has a singular zero of a finite degree k at $(m+1, t_0), \dots, (m+k, t_0)$. Then for all sufficiently small $\varepsilon > 0$ there exist open neighborhoods U^1 and U^2 of $u^1(0), u^2(0)$, such that for all $v^1(0), v^2(0) \in U^1 \times U^2$,*

$$(4.1) \quad \sum_{j=m+1}^{m+k} d_j(v^1 - v^2; -\varepsilon, +\varepsilon) \geq \frac{k}{2}.$$

Proof. By continuity and Lemma 1, (iii) there exists $\delta_0 > 0$ so that $u^1(t) - u^2(t)$ is not zero on $\{m, m+k+1\} \times [-\delta_0, \delta_0]$ and so that $u^1(t) - u^2(t)$ has no other

singular zeroes between 0 and $k + 1$ for $t \in [-\delta_0, \delta_0]$. By Lemma 2, (A3) and then (A1) and (A4), we see that for all $0 < \varepsilon \leq \delta_0$

$$(4.2) \quad \begin{aligned} z_{m+1, m+k}(u^1(\varepsilon) - u^2(\varepsilon)) &= z_{m+1, m+k}(u^1(-\varepsilon) - u^2(-\varepsilon)) \\ &= - \begin{cases} k-1 & \text{zero of Type I} \\ k & \text{zero of Type II} \end{cases} \leq -\frac{k}{2}. \end{aligned}$$

Now, given any $0 < \varepsilon \leq \delta_0$, as all the zeroes of $u^1(\pm\varepsilon) - u^2(\pm\varepsilon)$ are regular at $i = m + 1, \dots, m + k$, we can by continuity find neighborhoods U^1 and U^2 of $u^1(0)$ and $u^2(0)$ so that for any $v^1(0), v^2(0) \in U^1 \times U^2$,

$$\begin{aligned} z_{m+1, m+k}(u^1(\varepsilon) - u^2(\varepsilon)) &= z_{m+1, m+k}(v^1(\varepsilon) - v^2(\varepsilon)), \\ z_{m+1, m+k}(u^1(-\varepsilon) - u^2(-\varepsilon)) &= z_{m+1, m+k}(v^1(-\varepsilon) - v^2(-\varepsilon)), \end{aligned}$$

and so that $v^1(t) - v^2(t)$ is not zero on $\{m, m + k + 1\} \times [-\varepsilon, \varepsilon]$. The claim now follows from applying Lemma 2, (A3) and then (A1) to $v^2 - v^1$. \square

The zeroes of infinite degree however do not necessarily persist for small perturbations in the product topology, as a zero in a small neighborhood can "escape" towards $\pm\infty$. We can however show the following:

Lemma 6. *Assume $u(0), v(0) \in \mathcal{K}$ such that $u(0) - v(0)$ has a zero of degree ∞ . Then for each $\varepsilon > 0$ there exist $|t_0|, |t_1| < \varepsilon$ so that either $u(t_0) - v(t_1)$ or $u(t_0) - u(t_1)$, $u(t_0) \neq u(t_1)$ has a singular zero of a finite degree.*

The proof of Lemma 6 is simple but quite involved as it requires analysis of a number of cases with respect to the sign of derivatives $du^1(0)/dt$, $du^2(0)/dt$. We give it in Appendix 2.

4.3. The DC case. We now state analogues for the Lemmas above for du/dt instead of $u^1 - u^2$. Assume in the next two lemmas that F is DC.

Lemma 7. *Assume $u(0) \in \mathcal{K}$, such that du/dt has a singular zero of a finite degree k at $(m + 1, 0), \dots, (m + k, 0)$. Then for all sufficiently small $\varepsilon > 0$ there exist a neighborhood U of $u(0)$ such that for all $v(0) \in U$,*

$$\sum_{j=m+1}^{m+k} d_j(dv/dt; t_0 - \varepsilon, t_0 + \varepsilon) \geq \frac{k}{2}.$$

The proof is analogous to the proof of Lemma 5. Similarly, we can show (for the proof also see Appendix 2):

Lemma 8. *Assume $u(0) \in \mathcal{K}$ such that du/dt has a zero of degree ∞ . Then for each $\varepsilon > 0$ there exist $|t_0|, |t_1| < \varepsilon$ so that $u(t_0) - u(t_1)$, $u(t_0) \neq u(t_1)$ has a singular zero of a finite degree.*

5. THE AVERAGE NUMBER OF INTERSECTIONS AND ZEROES

This section is the core of the paper. We apply here the results of the previous section to solutions of (1.1), and define average number of self-intersections and derivative zeroes of a translationally invariant measure on \mathcal{X} , and average number of intersections of two probability measures on \mathcal{X} . We then show that these functions (denoted by Z) are weak Lyapunov, that is non-decreasing with respect to the evolution of (1.1). Furthermore, if there is a singular zero in the support of these measures, we show that Z is strictly decreasing. Finally, these properties are in some sense continuous in the weak topology of measures.

These ideas are measure-theoretical equivalents of the properties of the function z counting intersections of two solutions of (1.1) and (1.2) ([9]).

The intersection-counting function $(u, v) \mapsto z_{i,j}(u - v)$ is not invariant for the transformation R , thus not well defined on the quotient sets \mathcal{X}, \mathcal{K} . To fix that, we introduce two-argument functions $z_i, z_{i,j} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{N} \cup \{0\}$ with

$$\begin{aligned} z_i(u, v) &= \sum_{r \in \mathbb{Z}} z_i(u - v + r), \\ z_{i,j}(u, v) &= \sum_{r \in \mathbb{Z}} z_{i,j}(u - v + r). \end{aligned}$$

First note that $z_i, z_{i,j}$ are finite. Indeed, if $u, v \in \mathcal{K}_n$, then it is easy to see that $z_i(u, v) \leq 2n + 1$, thus $z_{i,j}(u, v) \leq (2n + 1)(j - i)$. Also $z_i, z_{i,j}$ are invariant for the transformation R in either argument, thus well defined on $\mathcal{X} \times \mathcal{X}$.

Our key object will be the space of T -invariant Borel probability measures $\mathcal{M}(\mathcal{K})$ on \mathcal{K} , and

$$\mathcal{M}(\mathcal{X}) = \bigcup_{k=1}^{\infty} \mathcal{M}(\mathcal{K}_n).$$

We will always consider the weak topology on $\mathcal{M}(\mathcal{K})$ and $\mathcal{M}(\mathcal{X})$ with respect to the product topology on \mathcal{X} . Note that \mathcal{K} is homeomorphic to a product of compact sets, thus compact. As \mathcal{K} is compact and T is continuous, $\mathcal{M}(\mathcal{K})$ is closed, and by the Tychonoff-Alaoglu theorem $\mathcal{M}(\mathcal{K})$ is compact.

Before proceeding, let us describe the space $\mathcal{M}(\mathcal{K})$. It is clearly non-empty (containing e.g. the Dirac measure δ_0 , where 0 is $u \equiv 0$), and actually very large. Most invariant measures of area-preserving twist maps generated by any generating function V are naturally embedded in $\mathcal{M}(\mathcal{K}_n)$ for sufficiently large n (see Section 2). We can also embed translationally invariant measures on $S^{\mathbb{Z}}$, where S is a bounded measurable subset of \mathbb{R} , in different ways in $\mathcal{M}(\mathcal{K})$.

The semiflow φ^t now naturally induces a continuous semiflow φ_*^t on $\mathcal{M}(\mathcal{K}_n)$. The flow φ^t on \mathcal{X} also induces a local flow φ_*^t on $\mathcal{M}(\mathcal{X})$, as one can easily see that one can find $\delta(n) > 0$ so that for all $u(0) \in \mathcal{K}_n$, $u(t) \in \mathcal{K}_{n+1}$ for $t \in [-\delta(n), 0]$ (e.g. by compactness argument). We will often use the notation $\mu(t) = \varphi_*^t \mu(0)$.

Given $\mu^1, \mu^2, \mu \in \mathcal{M}(\mathcal{K}_n)$, we define the intersection-counting functions

$$\begin{aligned} Z(\mu^1, \mu^2) &= \int \int z_0(u, v) d\mu^1(u) d\mu^2(v), \\ Z(\mu) &= Z(\mu, \mu), \\ \tilde{Z}(\mu) &= \int z_0\left(\frac{du}{dt}\right) d\mu(u), \end{aligned}$$

where du/dt is given by the right-hand side of (1.1). By definition,

$$\begin{aligned} 0 &\leq Z(\mu^1, \mu^2) \leq 2n + 1, \\ 0 &\leq \tilde{Z}(\mu) \leq 1. \end{aligned}$$

The function $Z(\mu^1, \mu^2)$ measures the average number of intersections with respect to two measures; the function $Z(\mu)$ measures the average number of "self-intersections" with respect to μ , and $\tilde{Z}(\mu)$ measures the average number of critical points $du_0/dt = 0$ with respect to (1.1). First note that the definitions above are invariant of the spatial location as μ^1, μ^2, μ are by definition T -invariant, thus the following holds for any integers $n < m$:

$$\begin{aligned} Z(\mu^1, \mu^2) &= \int z_n(u, v) d\mu^1(u) d\mu^2(v) \\ &= \frac{1}{m - n} \int z_{n,m}(u, v) d\mu^1(u) d\mu^2(v), \end{aligned}$$

analogously for $Z(\mu), \tilde{Z}(\mu)$.

We will see that Z, \tilde{Z} , are "almost" Lyapunov functions in the sense of LaSalle, thus they severely constrain the dynamics of φ_* . Generally the properties below will hold for functions Z in both AC and DC cases, and for the function \tilde{Z} in the DC case only.

Proposition 2. *If $\mu^1(0), \mu^2(0), \mu(0) \in \mathcal{M}(\mathcal{K})$, then the functions $t \mapsto Z(\mu^1(t), \mu^2(t))$, $t \mapsto Z(\mu(t))$ are non-increasing.*

Proof. We write shortly μ^1, μ^2 instead of $\mu^1(0), \mu^2(0)$. Let

$$(5.1) \quad Z(t) := Z(\mu^1(t), \mu^2(t)).$$

We now have for $0 \leq s < t$

$$Z(t) - Z(s) = \int \{z_0(u(t), v(t)) - z_0(u(s), v(s))\} d\mu^1(u(0)) d\mu^2(v(0)).$$

Applying Lemma 2, first the zero-balance equation (A1) for $m = 0, n = 1$, and then the T -invariance property (A2) of the crossing functions c_0, c_1 and the fact that μ^1, μ^2 are translationally invariant, we easily get

$$(5.2) \quad Z(t) - Z(s) = - \sum_{r \in \mathbb{Z}} \int d_0(u - v + r; s, t) d\mu^1(u(0)) d\mu^2(v(0))$$

which is ≤ 0 . The case $Z(\mu(t))$ follows by inserting $\mu^1 = \mu^2$. \square

The relation (5.2) will be very useful in the following, as it relates the occurrence of singular zeroes with the rate of decrease of Z . For future reference, because of translation invariance (Lemma 2, (A2)), for any $0 \leq s < t$, and integers $m < n$, (5.2) can be written as (using again shorthand (5.1))

$$(5.3) \quad Z(t) - Z(s) = - \frac{1}{n - m} \sum_{i=m}^{n-1} \sum_{r \in \mathbb{Z}} \int d_i(u - v + r; s, t) d\mu^1(u(0)) d\mu^2(v(0)).$$

We will say that two measures $\mu^1, \mu^2 \in \mathcal{M}(\mathcal{K})$ have a *non-transversal intersection*, if there exist u^1, u^2 in their supports so that $u^1 - u^2 + r$ has a singular zero for some $r \in \mathbb{Z}$. A non-transversal intersection is *proper*, if there is a singular zero as above with a finite degree.

We also say that any non-increasing function $t \rightarrow f(t)$ is strictly decreasing at t_0 , if for each $\varepsilon > 0$ there exists $\delta > 0$ so that $f(t_0 + \varepsilon) < f(t_0 - \varepsilon) - \delta$.

We now show that a proper non-transversal intersection implies that $Z(\mu^1, \mu^2)$ is strictly decreasing and that this also holds for small perturbations in the space of measures.

Proposition 3. *Assume $\mu^1(t_0), \mu^2(t_0) \in \mathcal{M}(\mathcal{K})$ have a proper non-transversal intersection. Then there exist neighborhoods U_1^*, U_2^* of $\mu^1(t_0), \mu^2(t_0)$ in $\mathcal{M}(\mathcal{K})$ so that for all $\nu^1(t_0) \in U_1^*, \nu^2(t_0) \in U_2^*$, $t \mapsto Z(\nu^1(t), \nu^2(t))$ is strictly decreasing at t_0 .*

Proof. Without loss of generality set $t_0 = 0$, and choose $u^1, u^2 \in K$ in the supports of $\mu^1(0), \mu^2(0)$ respectively, so that $u^1 - u^2$ has a singular zero of a finite degree $k \in \mathbb{N}$ at $(m, t_0), \dots, (m + k - 1, t_0)$. By Lemma 5, we find open neighborhoods U^1, U^2 of u^1, u^2 so that (4.1) holds for any $v^1(0), v^2(0) \in U^1 \times U^2$. As u^1, u^2 are in the supports of U^1, U^2 respectively, $\mu^1(U^1) \geq 2\delta_0, \mu^2(U^2) \geq 2\delta_0$ for some $\delta_0 > 0$.

Let U_1^*, U_2^* be the neighborhoods of $\mu^1(t_0), \mu^2(t_0)$ in $\mathcal{M}(\mathcal{K})$ so that for all $\nu^1 \in U_1^*, \nu^2 \in U_2^*$,

$$(5.4) \quad \nu^1(U^1) \geq \delta_0, \nu^2(U^2) \geq \delta_0.$$

Now if $Z(t) = Z(\nu^1(t), \nu^2(t))$, combining (5.3) (summing over $m+1, \dots, m+k$, with $t = \varepsilon$, $s = -\varepsilon$) with (4.1) we get

$$\begin{aligned} Z(\varepsilon) - Z(-\varepsilon) &\leq -\frac{1}{2} \int \mathbf{1}_{U^1}(\nu^1(0)) \mathbf{1}_{U^2}(\nu^2(0)) d\mu^1(\nu^1(0)) d\mu^2(\nu^2(0)) \\ &= -\frac{1}{2} \nu^1(U^1) \nu^2(U^2) \leq -\frac{1}{2} \delta_0^2, \end{aligned}$$

where $\mathbf{1}_U$ denotes the characteristic function of a set U . \square

We now extend this to functions $Z(\mu(t))$, $\tilde{Z}(\mu(t))$.

Proposition 4. *Assume $\mu(t_0) \in \mathcal{M}(\mathcal{K})$ has a proper non-transversal intersection with itself. Then there exists a neighborhood U^* of $\mu(t_0)$ in $\mathcal{M}(\mathcal{K})$ so that for all $\nu(t_0) \in U^*$, $t \mapsto Z(\nu(t))$ is strictly decreasing at t_0 .*

Proof. This follows from Proposition 3, by inserting $\mu = \mu^1 = \mu^2$ and taking $U^* = U_1^* \cap U_2^*$. \square

Proposition 5. *Assume F in (1.1) is DC. Choose $\mu(t_0) \in \mathcal{M}(\mathcal{K})$ with $u \in \mathcal{K}$ in its support such that du/dt has a singular zero of a finite degree. Then $t \mapsto \tilde{Z}(\mu(t))$ is strictly decreasing at t_0 .*

Proof. Analogously as Proposition 4, applying Lemma 7 instead of 5. \square

6. THE CHARACTERISTIC FLOW-MAP AND MAPS

This section is dedicated to the proofs of Theorems 1 and 2. Recall that a probability measure μ on \mathcal{K} is (φ, T) -invariant in the DC case, if it is invariant for all φ^t , $t \geq 0$ (time invariance) and for T (space invariance); and in the AC case if invariant for φ^1 and T . We say that a set $A \subseteq \mathcal{K}$ is (φ, T) -invariant, if it is in the DC case invariant for all φ^t , $t \geq 0$ and T ; and in the AC case invariant for φ^1 and T . Keeping in mind this, we will continue to treat the AC and DC cases simultaneously.

As ergodic theory for a commuting semiflow (in the AC case non-autonomous) and a homeomorphism is not standard, we give a few comments. First, it is not a-priori clear that (φ, T) -invariant measures exist on a compact, (φ, T) -invariant set such as \mathcal{K} . We address this in the next section, and construct a rich set of invariant measures.

A (φ, T) -invariant measure μ is ergodic, if for any Borel measurable (φ, T) -invariant set A , $\mu(A) \in \{0, 1\}$. An analogue to the ergodic decomposition theorem holds ([16], Proposition 4.1.12), so we will in some instances without loss of generality assume that $\mu \in \mathcal{M}(\mathcal{K})$ is ergodic. Note, for example, that given any ergodic measure $\mu \in \mathcal{M}(\mathcal{X})$, by ergodicity there is n large enough so that $\mu(\mathcal{K}_n) = 1$, thus $\text{supp}(\mu) \subseteq \mathcal{K}_n$ as \mathcal{K}_n is closed. Therefore assuming $\mu \in \mathcal{M}(\mathcal{K})$ rather than $\mu \in \mathcal{M}(\mathcal{X})$ is not a restriction.

Also note that, given a (φ, T) -ergodic measure, the Birkhoff averages with respect to φ^t or T do not necessarily μ -a.e. converge to the same value. The reason is that a (φ, T) -ergodic measure is not necessarily either φ -ergodic or T -ergodic.

First we show that φ is invertible on the support of any (φ, T) -invariant measure.

Lemma 9. *If μ is (φ, T) -invariant supported on \mathcal{K}_n for some $n \in \mathbb{N}$, then in the DC case $\varphi^t|_{\text{supp}(\mu)}$ is a continuous flow, and in the AC case $\varphi^1|_{\text{supp}(\mu)}$ is a homeomorphism.*

Proof. As $\text{supp}(\mu)$ is (φ, T) -invariant and φ is continuous, it is easy to see that for any $t \geq 0$ (in the AC case t an integer), $\varphi^t(\text{supp}(\mu)) = \text{supp}(\mu)$. By Lemma 4, φ is a local flow, thus injective. We conclude that φ^t is bijective on $\text{supp}(\mu)$. As

\mathcal{X} is metrizable (a quotient space of a metric space, with metric induced by any norm (3.3)) and \mathcal{K}_n is compact, $\text{supp}(\mu)$ is compact and $\varphi^t : \text{supp}(\mu) \rightarrow \text{supp}(\mu)$ is a homeomorphism for all t . \square

Proof of Theorem 1. Assume $\mu \in \mathcal{M}(\mathcal{K})$ is (φ, T) -invariant. First we show that $t \mapsto Z(\mu(t))$ (where $\mu(0) = \mu$) is constant. In the DC case, this must be true as $\mu(t) = \mu(0)$ for all $t \geq 0$. In the AC case, this follows from $\mu(n) = \mu(0)$ for integer n , and Lemma 2 which showed that $t \mapsto Z(\mu(t))$ is non-decreasing.

It is clear from the definition that the map $\pi : \mathcal{K} \rightarrow \mathbb{T} \times \mathbb{R}$ defined with (1.4) is continuous. We first show that $\pi|_{\text{supp}(\mu)}$ is injective. Indeed if $u^1, u^2 \in \text{supp}(\mu)$ are such that $\pi(u^1) = \pi(u^2)$, then $u^1 - u^2 + r$ has a singular zero of a finite or infinite degree for some $r \in \mathbb{Z}$. If the degree is finite, by Proposition 4, $t \mapsto Z(\nu(t))$ is strictly decreasing at $t = 0$, where $\nu(0) = \mu$, which is a contradiction. Now if the degree of the zero is infinite in the DC case, by Lemma 6 and φ -invariance of μ , one can find other $v^1, v^2 \in \text{supp}(\mu)$ so that $v^1 - v^2 + r$ has a singular zero of a finite degree, which reduces it to the previous case.

Consider the case of a singular zero of infinite degree in the AC case. By Lemma 6, we find $v^1(t_0) - v^2(t_1)$ with a singular zero of a finite degree such that $v^1(0), v^2(0) \in \text{supp}(\mu)$. By the same argument as above, we show that $t \mapsto Z(\mu(t + t_0), \mu(t + t_1))$ is non-decreasing with period 1, thus constant, which is a contradiction with Proposition 3 and $v^1(t_0) \in \text{supp}(\mu(t_0)), v^2(t_1) \in \text{supp}(\mu(t_1))$.

Now assume the contrary, i.e. that there are two (φ, T) -invariant measures $\mu^1, \mu^2 \in \mathcal{M}(\mathcal{K})$, so that for some $u^1 \in \text{supp}(\mu^1), u^2 \in \text{supp}(\mu^2)$, we have $\pi(u^1) = \pi(u^2)$. But $\mu = \mu^1/2 + \mu^2/2$ is also a (φ, T) -invariant measure whose support contains both u^1, u^2 , which is impossible.

Finally, the case $\mu \in \mathcal{M}(\mathcal{X})$ reduces by the ergodic decomposition theorem to analysis on $\mathcal{M}(\mathcal{K}_n)$ for sufficiently large n . \square

Remark 1. *We actually proved more in the proof above: if $u, v \in \mathcal{A}$, then $u - v$ can not have a singular zero.*

Proof of Theorem 2. By definition, $\pi : \mathcal{K} \rightarrow \tilde{\mathcal{K}}$ is continuous and by Theorem 1 it is bijective. As \mathcal{K} is compact and $\tilde{\mathcal{K}}$ a metric space, $\pi : \mathcal{K} \rightarrow \tilde{\mathcal{K}}$ is a homeomorphism. The claim now follows from Lemma 9 and definition (1.5) of $\tilde{T}, \tilde{\varphi}$. \square

We complete the Section with a conjecture and a couple of comments on the likely structure of the characteristic map \tilde{T} associated to (1.1).

Conjecture 1. *The map \tilde{T} can be canonically extended to a homeomorphism $\tilde{T} : \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{T}^1 \times \mathbb{R}$, diffeomorphic to an area-preserving twist diffeomorphism $f_{V,F}$.*

The numerical simulations [7] and our construction suggest the following description of \tilde{T} . Given an AC forcing F , we can define

$$\begin{aligned} \overline{F} &= \int_0^1 F(t) dt, \\ \sigma(F) &= \left(\int_0^1 (F(t) - \overline{F})^2 dt \right)^{1/2}. \end{aligned}$$

We conjecture that for a fixed $V, \sigma(F)$ (including the DC case $\sigma(F) = 0$), there is F_c such that for all $\overline{F} \geq F_c$, the map $f_{V,F}$ is integrable. Conversely, for a fixed V, \overline{F} , the limit $\sigma(F) \rightarrow \infty$ corresponds to an "anti-integrable limit" of the family of maps $f_{V,F}$.

7. THE AUBRY-MATHER THEOREM

This section is dedicated to the proof of Theorem 3. We first define rotationally ordered configurations in $\mathbb{R}^{\mathbb{Z}}$ and the rotation number and deduce their main properties, then show in general existence of (φ, T) -invariant measures, and from this complete the proof.

Recall the definitions of translations $T(u)_i = u_{i+1}$ and $Ru = u + 1$ commuting with (1.1) and each other. We can define $T_{p,q} = T^p R^q$, or $T_{p,q}(u)_i = u_{i+p} + q$ for any $p, q \in \mathbb{Z}$. We say that a configuration $u \in \mathbb{R}^{\mathbb{Z}}$ is rotationally ordered, if for any $p, q \in \mathbb{Z}$, either $T_{p,q}u \leq u$ or $T_{p,q}u \geq u$ (i.e. the set of all $T_{p,q}$ translations is totally ordered).

The following is an elementary result of Aubry-Mather theory; for a proof see e.g. Bangert [6], Corollary 3.2:

Lemma 10. *If $u \in \mathbb{R}^{\mathbb{Z}}$ is a rotationally ordered configuration, then there exists a unique $\rho \in \mathbb{R}$ so that for all $i, j \in \mathbb{Z}$,*

$$(7.1) \quad |u_j - u_i - \rho(j - i)| \leq 1.$$

The number ρ can be characterized as $\rho = \sup\{p/q : T_{p,q}u \leq u\} = \inf\{p/q : T_{p,q}u \geq u\}$.

If $u \in \mathbb{R}^{\mathbb{Z}}$ is such that the limit

$$\rho(u) = \lim_{|j-i| \rightarrow \infty} \frac{u_j - u_i}{j - i}$$

exists, we call it the rotation number (or mean spacing) of a configuration. For rotationally ordered configurations, ρ in (7.1) is its rotation number.

Now we show existence of (φ, T) -invariant measures.

Lemma 11. *Assume $\mathcal{R} \subset \mathcal{K}$ is a closed, non-empty (φ, T) -invariant set. Then there exists a (φ, T) -invariant probability measure supported on \mathcal{R} .*

Proof. Choose any Borel probability measure ν on \mathcal{R} (say Dirac supported on any element). We in the DC case define

$$\nu^n = \frac{1}{n^2} \int_0^n \left(\sum_{k=1}^n \varphi_*^t T_*^k \nu \right) dt,$$

and in the AC case

$$\nu^n = \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \varphi_*^j T_*^k \nu dt.$$

Here φ_*^t, T_*^k is pulling of measures with respect to φ^t, T^k . As \mathcal{R} is compact and ν^n are well defined probability measures supported on \mathcal{R} , ν^n has a weakly convergent subsequence $\nu^{n_k} \rightarrow \mu$ and μ is supported on \mathcal{R} . It is easy to check that μ is (φ, T) -invariant (e.g. by checking this for continuous functions $f : \mathcal{R} \rightarrow \mathbb{R}$ and then applying the Riesz representation Theorem). \square

Proof of Theorem 3. Denote by \mathcal{R}_ρ the set of all rotationally ordered configurations with the rotation number ρ . It is non-empty, as u defined with $u_i = \rho i$ is in \mathcal{R}_ρ . From (7.1) one immediately sees that $\mathcal{R}_\rho \subset \mathcal{K}_n$ where n is any integer $\geq |\rho| + 1$. From the definition it follows that \mathcal{R}_ρ is closed, thus it is compact. As $T_{p,q}$ commutes with φ and T , by Lemma 3 φ is order preserving and T is trivially order preserving, we see that \mathcal{R}_ρ is (φ, T) -invariant. It is now enough to apply Lemma 11 to \mathcal{R}_ρ . \square

8. DESCRIPTION OF INVARIANT MEASURES IN THE DC CASE

Here we prove Theorem 4, and so describe the set of (φ, T) -invariant measures on \mathcal{X} and \mathcal{K} , denoted by $\mathcal{E}(\mathcal{X})$ and $\mathcal{E}(\mathcal{K})$, in the DC case. The standing assumption in this section is that F in (1.1) is DC. We first define the average speed of a measure and a configuration. We then show by applying the Poincaré-Bendixson theorem that any $\mu \in \mathcal{E}(\mathcal{K})$ is entirely supported on equilibria and periodic orbits. If the average speed is $\neq 0$, then μ is supported on a single periodic orbits. Finally, if the average speed is zero, we show that μ is supported on equilibria.

Let $\mu \in \mathcal{E}(\mathcal{X})$ and $u \in \mathcal{X}$. We can define their average speeds $v(\mu)$, $v(u)$ as

$$(8.1) \quad v(\mu) = \int (u_0(1) - u_0(0)) d\mu,$$

$$(8.2) \quad v(u) = \lim_{t \rightarrow \infty} \frac{u_0(t) - u_0(0)}{t},$$

where $v(u)$ is defined when (8.2) is convergent.

Lemma 12. *If μ is (φ, T) -ergodic on \mathcal{X} , then for μ -a.e. u , $v(u) = v(\mu)$.*

Proof. As μ is φ -invariant (but not necessarily φ -ergodic!), there is a set of full measure such that $v(u)$ is well defined. As by definition and Lemma 4 the spacing $|u_{i+1}(t) - u_i(t)|$ is bounded uniformly in i, t , the definition (8.2) is independent of i in u_i , thus $v(u) = v(Tu)$. Thus $v(u)$ is defined on a set of full measure, φ and T -invariant measurable function, so it must be constant μ -a.e. and equal to $v(\mu)$. \square

We will use a version of the well-known Poincaré-Bendixson theorem on asymptotics of 2-dimensional flows. For a general discussion of the Poincaré-Bendixson theorem see e.g. [16], Section 14.1; for the proof of the exact formulation below see [9], the proof of Proposition 1 (the only difference being that they consider compact subsets of the plane, which makes no difference, as the argument works for any compact sets which can be embedded in a two dimensional sphere).

Lemma 13. *Assume \tilde{K} is a compact subset of the cylinder $\mathbb{T}^1 \times \mathbb{R}$, and $\tilde{\varphi}$ a complete continuous flow on \tilde{K} generated by a C^0 vector field on \tilde{K} . Then the only recurrent points of $\tilde{\varphi}$ are equilibria and periodic orbits.*

We need the next lemma to be able to claim that if $\pi(u)$ is an equilibrium, then u must be an equilibrium.

Lemma 14. *If $u \in \mathcal{A}$, then $w = du/dt$ can not have a singular zero.*

Proof. Find a measure $\mu \in \mathcal{E}(\mathcal{K})$ for some $\mathcal{K} = \mathcal{K}_n$ so that u is in the support of μ . As $\mu = \mu(0)$ is by definition (φ, T) -invariant, $t \mapsto \tilde{Z}(\mu(t))$ is a constant function. Now if $w = du/dt$ has a singular zero of a finite degree, from Proposition 5 we obtain that $\tilde{Z}(\mu(t))$ is strictly decreasing at $t = 0$, which is a contradiction. If w has a singular zero of infinite degree, by Lemma 8 we can find v^1, v^2 in the support of μ so that $v^1 - v^2$ has a singular zero of finite degree. This implies that by Proposition 3, $t \mapsto Z(\mu(t))$ is strictly decreasing at $t = 0$, which is again a contradiction with μ being (φ, T) -invariant. \square

We now show that all of this combined with our results on the projection π gives a fairly good description of supports of $\mu \in \mathcal{E}(\mathcal{X})$.

Lemma 15. *Assume μ is (φ, T) -ergodic. Then μ is supported on φ -periodic orbits with the same period $t_0 > 0$, or on equilibria.*

Proof. As μ is (φ, T) -ergodic, there exists $\mathcal{K} = \mathcal{K}_n$ for n large enough so that $\mu \in \mathcal{E}(\mathcal{K})$. Take \tilde{K} , $\tilde{\varphi}$ as in (1.5). Now we deduce from Theorems 1 and 2 that most of the assumptions of Lemma 13 are satisfied; we only need to show that $\tilde{\varphi}$ is generated by a C^0 vector field. One can easily check that for $(x, p) \in \mathbb{T}^1 \times \mathbb{R}$, the vector field is given by

$$(8.3) \quad (x, p) \mapsto \begin{bmatrix} d(\pi^{-1}(x, p))_0/dt \\ d(\pi^{-1}(x, p))_1/dt - d(\pi^{-1}(x, p))_0/dt \end{bmatrix},$$

where du_0/dt , du_1/dt for $u = \pi^{-1}(x, p)$ are defined with the right-hand side of (1.1). From Lemma 13 we deduce that all $\tilde{\varphi}$ -recurrent orbits on \tilde{K} are periodic orbits and equilibria. Consider the pulled $\tilde{\varphi}$ -invariant measure $\pi_*\mu$ on \tilde{K} . As \tilde{K} is metrizable and compact, the set of $\tilde{\varphi}$ -recurrent points has full measure with respect to $\pi_*\mu$ (see e.g. [16], the proof of Proposition 4.1.18). Now note that by Theorem 1, π^{-1} maps periodic orbits into periodic orbits, and by Lemma 14, π^{-1} maps equilibria into equilibria (otherwise the right-hand side of (8.3) would be zero, i.e. for $u = \pi^{-1}(x, p)$, du/dt would have a singular zero, which is impossible by Lemma 14). If we denote by \mathcal{P}_t the orbits in \mathcal{K} with period t , $t \geq 0$ (where \mathcal{P}_0 are equilibria), we now have

$$(8.4) \quad \mu \left(\bigcup_{t \geq 0} \mathcal{P}_t \right) = 1.$$

Now for each $t \geq 0$, \mathcal{P}_t is (φ, T) -invariant, thus by ergodicity $\mu(\bigcup_{t \leq t_0} \mathcal{P}_t) \in \{0, 1\}$. As $t_0 \mapsto \mu(\bigcup_{t \leq t_0} \mathcal{P}_t)$ is non-decreasing, there exists a unique t_0 so that $\mu(\mathcal{P}_{t_0}) = 1$. As \mathcal{P}_{t_0} is closed, it contains the support of μ . \square

We finally need to show that, if μ is supported on periodic orbits and has non-zero speed, it is supported on a single one.

We say that the orbit of u is totally ordered, if the set $\Upsilon(u) = \{\varphi^t T^k u, t \in \mathbb{R}, k \in \mathbb{Z}\}$ is totally ordered. That means that for any $v, w \in \Upsilon(u)$, we have either $v \geq w$ or $v \leq w$.

Lemma 16. *Assume μ is (φ, T) -ergodic, $v(\mu) \neq 0$, and supported on periodic orbits with period $t_0 > 0$. Then it is supported on a single totally ordered periodic orbit such that the $\pi \circ T^j$ -image of $\text{supp}(\mu)$ is independent of $j \in \mathbb{Z}$.*

Proof. We prove it in three steps. First we show that $u(t_0) = u(0) \pm 1$, then we show that $u(0)$ and $u(t)$ in the support of μ can not intersect, and finally we show uniqueness of the periodic orbit.

The average non-zero speed is by Lemma 12 the same for all the periodic orbits in $\text{supp}(\mu)$. By Theorem 1, π projects each of them to a homotopically non-trivial invariant circle on $\mathbb{T}^1 \times \mathbb{R}$ (otherwise the speed would be zero), thus $u(t_0) = u(0) \pm 1$, with the sign the same as the sign of $v(\mu)$. Assume without loss of generality $u(t_0) = u(0) + 1$.

Now assume $u(0)$ and $u(t_1)$ intersect, $0 < t_1 < t_0$ (the only analogous alternative is $-t_0 < t_1 < 0$). Consider $w(t) = u(t) - u(0)$. Recall that by Theorem 1 and Remark 1, all intersections in the support of μ are transversal, so $w(t)$ has a regular zero, say at i . Then $w(t)$ intersects zero at

$$(8.5) \quad x(t) = \frac{(i+1)u_i - i(u_{i+1})}{u_i - u_{i+1}}.$$

As $w(t)$ has no singular zeroes, it is easy to see we can continue the function $t \mapsto x(t)$ locally with (8.5) if $x(t)$ not an integer, and if $x(t) = i$ an integer, with

$$x(t) = \begin{cases} \frac{(i+1)u_i - i(u_{i+1})}{u_i - u_{i+1}} & u_i(t) \geq 0, \\ \frac{i u_{i-1} - (i-1)u_i}{u_{i-1} - u_i} & u_i(t) < 0. \end{cases}$$

Thus we can find a continuous function $t \mapsto x(t)$ so that $w(t)$ has a regular zero at $\lfloor x(t) \rfloor$. But by assumptions $w(t_0) \equiv 1$, so the only possibility is that for some t_2 , $t_1 < t_2 < t_0$, $\lim_{t \rightarrow t_2^-} x(t) = +\infty$ or $\lim_{t \rightarrow t_2^-} x(t) = -\infty$. Assume without loss of generality $\lim_{t \rightarrow t_2^-} x(t) = +\infty$. As by the proof of Lemma 4, $t \mapsto w(t)$ is continuous in l_∞ , we easily see that $\lim_{i \rightarrow \infty} w_i(t_2) = 0$, thus

$$(8.6) \quad \lim_{i \rightarrow \infty} |u_i(t_2) - u_i(0)| = 0.$$

By compactness, we can find a sequence T^{n_k} , $n_k \rightarrow \infty$ such that $T^{n_k}u(0)$ converges to some v . As T commutes with φ , from (8.6) we see that $v(t_2) = v(0)$, and v is in the support of μ . But $t_2 < t_0$, which is in contradiction with the assumption.

Now we see that each periodic orbit in the π -image of support of μ intersects each x coordinate exactly once at a point p , where $(x, p) \in \mathbb{T}^1 \times \mathbb{R}$. We can thus identify the quotient set $\text{supp}(\mu)/\varphi$ with a compact set \bar{B} of intersections at $x = 0$, and consider \bar{B} to be a subset of \mathbb{R} parametrized with p . By definitions, T induces a homeomorphism \bar{T} on \bar{B} , and the induced flow $\bar{\varphi}$ is constant on \bar{B} . The induced measure $\bar{\mu}$ is $(\bar{T}, \bar{\varphi})$ -ergodic on \bar{B} , and as $\bar{\varphi}$ is constant, it must be \bar{T} ergodic. Finally, note that \bar{T} is order preserving on \bar{B} , that means that $p > q$ implies $\bar{T}(p) > \bar{T}(q)$. We see that, as otherwise we find u, v in equivalence classes of p, q which violate order preserving, such that $u_0(t) = v_0(t')$, and then $u(t) - v(t')$ has a singular zero at 0. As by assumptions, \bar{B} is the support of $\bar{\mu}$, \bar{T} a homeomorphism and $\bar{\mu}$ is \bar{T} -ergodic, there must be $p_0 \in \bar{B}$ such that its ω -limit set with respect to \bar{T} is the entire \bar{B} . But $\bar{T}^n(p_0)$ is monotone and bounded, thus converges to a single point. We conclude that \bar{B} must be a single point, so support of μ consists of a single periodic orbit.

It must be a totally ordered set, as we have shown that $u(t), u(t')$ do not intersect unless equal. \square

Proof of Theorem 4. Take any (φ, T) -ergodic measure μ on \mathcal{X} . If $v(\mu) = 0$, consider the function $\int V(u_0, u_1) d\mu$. The derivatives of V and u_i are by compactness uniformly bounded on the semiorbit of $u(t)$. Thus we can exchange the integral and derivation, use T -invariance of μ , (1.1) and finally φ -invariance of μ and obtain

$$\begin{aligned} \frac{d}{dt} \int V(u_0, u_1) d\mu &= \int (V_1(u_0, u_1)u'_0(t) + V_2(u_0, u_1)u'_1(t)) d\mu \\ &= \int (V_1(u_0, u_1)u'_0(t) + V_2(u_{-1}, u_0)u'_0(t)) d\mu \\ &= - \int (u'_0(t))^2 d\mu + F \int u'_0(t) d\mu \\ &= - \int (u'_0(t))^2 d\mu + F \int \left(\int_0^1 u'_0(t) dt \right) d\mu \\ (8.7) \quad &= - \int (u'_0(t))^2 d\mu + F v(\mu). \end{aligned}$$

However V is continuous on \mathcal{X} and μ is φ -invariant, so the left-hand side of (8.7) must be 0. We deduce that for μ -a.e. u , $u'_0(t) = 0$, so by continuity argument and T -invariance we see that the support of μ consists of equilibria.

If $v(\mu) \neq 0$, the claim follows from lemmas 15 and 16. \square

Remark 2. *The relation*

$$F \cdot v(\mu) = \int (u'_0(t))^2 d\mu$$

obtained above, thus valid for DC dynamics, has standard physical interpretations, such as force \times speed=energy dissipation, or voltage \times current=resistance.

9. UNIFORMLY SLIDING STATES IN THE DC CASE

In this section we further discuss the DC case when the average speed is not zero. We show that the periodic orbit is then a uniformly sliding state, and discuss the properties of its "modulation function". The results of this section have already been obtained by Baesens, MacKay and Qin ([4], [5], [22], [23]), by combination of a limiting procedure (typically going from rational to irrational rotation numbers) and a fixed point argument.

We show that these results follow directly from our ergodic-theoretical and topological considerations, hoping for extension of these results to AC driving and systems similar to (1.1).

We say that a solution of (1.1) is a uniformly sliding solution with the rotation number ρ , speed v and time shift α , if there exists a modulation function $m : \mathbb{R} \rightarrow \mathbb{R}$, $m(x+1) = m(x)$, such that for each $j \in \mathbb{Z}$, $t \in \mathbb{R}$,

$$u_j(t) = j\rho + vt + \alpha + m(j\rho + vt + \alpha).$$

Corollary 1. *Assume that F is DC, μ is (φ, T) -ergodic and $v(\mu) \neq 0$. Then μ is supported on a single uniformly sliding solution.*

Proof. Assume without loss of generality $v(\mu) > 0$. As by Lemma 16 and Theorems 1, 2, π projects $u_j(t)$ to a single periodic orbit with period t_0 independently of j and with the same projected flow $\tilde{\varphi}$, there exists a function $\tilde{m} : \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha_j \in \mathbb{R}$ such that $\tilde{m}(t+t_0) = \tilde{m}(t) + 1$ and for all $j \in \mathbb{Z}$, $u_j(t) = \tilde{m}(t + \alpha_j)$. It is easy to see that $v = v(\mu) = 1/t_0$. We set $m(t) := \tilde{m}(t \cdot t_0) - t$, and thus $m(t+1) = m(t)$ and

$$(9.1) \quad u_j(t) = m(vt + v\alpha_j) + vt + v\alpha_j.$$

As the set $\{u(t), t \in \mathbb{R}\}$ is by Theorem 4 totally ordered, $u(t)$ is rotationally ordered, so by Lemma 10, (7.1) must hold with some $\rho = \rho(u(t))$. As $t \mapsto u(t) - u(0)$ is continuous in $l_\infty(\mathbb{Z})$, all $u(t)$ must have the same rotation number equal to the rotation number of the measure defined as

$$\begin{aligned} \rho(\mu) &: = \int (u_1 - u_0) d\mu \\ &= v \int \left(\int_0^{t_0} (u_{i+1}(t) - u_i(t)) dt \right) d\mu, \end{aligned}$$

where in the second row we used T and φ -invariance of μ . By Lemma 16, $\int_0^{t_0} (u_{i+1}(t) - u_i(t))$ is independent of i as it is the second coordinate of the projection π integrated over one period, and by periodicity independent of $u \in \text{supp}(\mu)$, thus for all $i \in \mathbb{Z}$, $t \in \mathbb{R}$,

$$(9.2) \quad \int_0^{t_0} (u_{i+1}(t) - u_i(t)) dt = \frac{\rho}{v}.$$

Combining (9.1) and (9.2) we get $v\alpha_j = j\rho + \alpha$ for some $\alpha \in \mathbb{R}$, which completes the proof. \square

From Theorems 3, 4 now follow the results from [4], [22] that if there is no equilibrium with a given rotation number $\rho \in \mathbb{R}$, there must be an uniformly sliding solution with the same rotation number. Furthermore, one can prove continuous interdependence of v, ρ, F and other parameters of (1.1) by considering continuity of (φ, T) -invariant measures and Theorems 3, 4.

10. THE SPATIO-TEMPORAL ATTRACTOR

In this section we prove Theorem 5. We need to show that, given any T -invariant probability measure μ on \mathcal{X} and any open set neighbourhood U of \mathcal{X} , (1.6) holds. As \mathcal{K}_n are closed and T -invariant, by the ergodic decomposition theorem it is sufficient to prove that for $\mu \in \mathcal{M}(\mathcal{K}_n)$ (denoted again with $\mathcal{M}(\mathcal{K})$).

Recall the definition of $P_{\mu, S}$ in Section 2. It is easy to see that the left-hand side of (1.6) can be written as

$$(10.1) \quad P_{\mu, S}(\{(u, t), \varphi^t(u) \in A\}) = \frac{1}{S} \int_0^T (\varphi_*^t \mu(A)) dt.$$

Denote the probability measure on the right-hand side of (10.1) with ν^S . It is easy to see that $\nu^S \in \mathcal{M}(\mathcal{K})$. We want to show that, given any open neighbourhood U of \mathcal{A} , then $\lim_{S \rightarrow \infty} \nu^S(U) = 1$. Assume the contrary, and find a sequence $S_k \rightarrow \infty$ and $\delta > 0$ so that $\nu^{S_k}(U) < 1 - \delta$ for all k . As $\mathcal{M}(\mathcal{K})$ is compact, ν^{S_k} has a subsequence for simplicity denoted also with ν^{S_k} , which converges to some $\mu \in \mathcal{M}(\mathcal{K})$. It is easy to deduce from (10.1) that μ is also φ -invariant. Then μ must be supported on \mathcal{A} , thus $\mu(\mathcal{A}) = 1$, $\mu(U) = 1$ and as U open, $\lim_{k \rightarrow \infty} \nu^{S_k}(U) = 1$ which is a contradiction.

11. APPENDIX 1: ON THE ZERO-SET OF A SYSTEM OF INFINITELY MANY COOPERATIVE EQUATIONS

11.1. The proof of Lemma 1. Recall the setting and definitions from Section 3, and assume in the following all the assumptions of Lemma 1.

Lemma 17. *Assume $w_1(0), \dots, w_k(0) = 0$ is a singular zero of degree k . Then there exist real numbers d_1, \dots, d_k such that for all $j = 1, \dots, k$,*

$$(11.1) \quad w_j(t) = d_j t^{j^*} + o(t^{j^*}),$$

where $j^* = \min\{j, k+1-j\}$.

Proof. We prove inductively in l , $1 \leq l \leq (k+1)/2$, that for all j , $l \leq j \leq k+1-l$,

$$(11.2) \quad w_j(t) = d_{j,l} \cdot t^l + o(t^l).$$

For $l = 1$ and all $1 \leq j \leq k$, as w_j is C^1 and $w_j(0) = 0$, by the Taylor formula,

$$w_j(t) = \frac{dw_j(0)}{dt} t + o(t),$$

thus

$$(11.3) \quad d_{j,1} = dw_j(0)/dt.$$

We now assume (11.2) holds for some $l-1$. Inserting (11.2) in the right-hand side of (3.1) and using (3.2) (continuity of a_j, b_j, c_j) we get

$$\begin{aligned} \frac{d}{dt} w_j(t) &= (a_j(t) d_{j-1, l-1} + b_j(t) d_{j+1, l-1} + c_j(t) d_{j, l-1}) \cdot t^{l-1} + o(t^{l-1}) \\ &= (a_j(0) d_{j-1, l-1} + b_j(0) d_{j+1, l-1} + c_j(0) d_{j, l-1}) t^{l-1} + o(1) t^{l-1} + o(t^{l-1}) \\ (11.4) \quad &= (a_j(0) d_{j-1, l-1} + b_j(0) d_{j+1, l-1} + c_j(0) d_{j, l-1}) t^{l-1} + o(t^{l-1}). \end{aligned}$$

Integrating it and using $w_j(0) = 0$ we obtain (11.2) for l and complete the inductive step with

$$(11.5) \quad d_{j,l} = \frac{1}{l} (a_j(0)d_{j-1,l-1} + b_j(0)d_{j+1,l-1} + c_j(0)d_{j,l-1}).$$

□

We now need to show that $d_j \neq 0$ and determine their sign. Denote by $\text{sgn}(x) \in \{-1, 0, 1\}$ the sign of $x \in \mathbb{R}$.

Lemma 18. *Assume all as in Lemma 17. Then for all $j = 1, \dots, k$, except in the case $j = k/2$, k even, the zero of Type II, we have*

$$(11.6) \quad \text{sgn}(d_j) = \text{sgn}(w_b(0)),$$

where $b \in \{0, k+1\}$ such that $|j - b| \leq |j - (k+1-b)|$.

Proof. From (11.3) and (3.1) we get

$$\begin{aligned} d_{1,1} &= dw_1(0)/dt = a_1(0)w_0(0), \\ d_{k,1} &= dw_k(0)/dt = b_k(0)w_{k+1}(0), \\ d_{j,1} &= 0 \text{ for } j = 2, \dots, k-1. \end{aligned}$$

Iteratively from (11.5) we easily get using again $j^* = \min\{j, k+1-j\}$

$$\begin{aligned} d_j &= w_0(0) \prod_{i=1}^j a_i(0)/j^*! && \text{for } j < k/2, \\ d_j &= w_{k+1}(0) \prod_{i=j}^k b_i(0)/j^*! && \text{for } j > k/2, \\ d_j &= w_0(0) \prod_{i=1}^j a_i(0)/j^*! + w_{k+1}(0) \prod_{i=j}^k b_i(0)/j^*! && \text{for } j = k/2, \end{aligned}$$

which completes the proof. □

Relations (11.1) and (11.6) now complete the description of the behavior of a solution around a singular zero of degree k for small t . We summarize it, again assuming without loss of generality (i.e. up to reparametrization) that $(1, 0), \dots, (k, 0) \in S$ is a zero of degree k at $t = 0$. In that case, lemmas 17 and 18 imply that, assuming all as in Lemma 17, that there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, we have (see Figure 1):

Type of zero	k	$z_{0,k+1}(w(-\delta))$	$z_{0,k+1}(w(0))$	$z_{0,k+1}(w(\delta))$
I	even	$k+1$	k	1
I	odd	k	k	1
II	even	k	k	0
II	odd	$k+1$	k	0



Figure 1: Types of singular zeroes of finite degree

We conclude that the number of zeroes is strictly decreasing at singular zeroes. By continuity, the number of zeroes does not change at regular zeroes, which completes the proof of Lemma 1.

11.2. Zeroes of degree ∞ . As noted earlier, zeroes of degree ∞ require special attention.

Lemma 19. *Assume all as in Lemma 1. Suppose $w(0)$ has a zero of degree ∞ , $w_0(0) \neq 0$ and $w_i(0) = 0$ for all $i \geq 1$. Then there exist d_j having the same sign as $w_0(0)$ such that for all $j \geq 1$,*

$$(11.7) \quad w_j(t) = d_j t^j + o(t^j).$$

Proof. An analogous analysis as in the proofs of Lemmas 17 and 18 easily yields (11.7) with

$$d_j = \frac{w_0(0)}{j!} \prod_{i=1}^j a_i(0).$$

□

11.3. Proof of Lemma 2. We construct explicitly the functions $c_i(s, t)$, $d_i(s, t)$. We first define sets of zeroes S^+ , S^- which are subsets of the set of singular zeroes $S \subset \mathbb{Z} \times [T_0, T_1]$ which "disappear" either at $t = t_0$ or at $t = t_0^+$. Assume without loss of generality that $(1, t_0), \dots, (k, t_0)$ is a singular zero of degree k . Then we define:

(i) Zero of Type I, k even, zero at $i = 1, \dots, 2l = k$:

$$\begin{aligned} S^- &\ni (1, t_0), \\ S^+ &\ni (1, t_0), (2, t_0), \dots, (l-1, t_0), (l+1, t_0), \dots, (2l, t_0). \end{aligned}$$

(ii) Zero of Type I, k odd, zero at $i = 1, \dots, 2l+1 = k$:

$$S^+ \ni (1, t_0), (2, t_0), \dots, (l, t_0), (l+2, t_0), \dots, (2l+1, t_0).$$

(iii) Zero of Type II, k even, zero at $i = 1, \dots, 2l = k$:

$$S^+ \ni (1, t_0), \dots, (2l, t_0).$$

(iv) Zero of Type II, k odd, zero at $i = 1, \dots, 2l+1 = k$:

$$\begin{aligned} S^- &\ni (l+1, t_0), \\ S^+ &\ni (1, t_0), \dots, (2l+1, t_0). \end{aligned}$$

(v) Zero of degree ∞ at $i \geq 1$:

$$S^+ \ni (1, t_0), (2, t_0), \dots$$

Also S^-, S^+ contain no other elements apart from those defined above. We now define

$$d_i(s, t) = |S^+ \cap \{i\} \times [s, t]| + |S^- \cap \{i\} \times (s, t]|,$$

where $|A|$ is the cardinal number of a set. The rationale for this is clear from Figure 1 and (11.1), (11.7).

Without loss of generality we construct $c_0(0, S)$ for some $S > 0$. For each $t \in [0, S]$ we choose an open interval $I_t \subset \mathbb{R}$ containing t in the following way:

(i) If $w_0(t) \neq 0$, then I_t is such that for all $s \in I_t$, $w_0(s) \neq 0$.

(ii) If $w_0(t) = 0$, a regular zero, then I_t is such that for all $s \in I_t$, $w_{i-1}(s) \neq 0$, $w_{i+1}(s) \neq 0$.

(iii) If $w_0(t) = 0$ is a singular zero of Type I, degree odd, zero at $-l, -l+1, \dots, l$ (i.e centered at 0), we choose I_t so that for all $s \in I_t$, $s \neq t$, we have $u_j(s) \neq 0$ for $j = -l, \dots, -1, 1, \dots, l$.

(iv) If $w_0(t) = 0$ is a singular zero in all other cases, I_t is such that for all $s \in I_t$, $s \neq t$, we have $w_0(s) \neq 0$.

We constructed the neighborhoods I_t in such a way that for each time $s \in I_t$, there is at most one zero which may be crossing $i = 0$ (the singular zero cases follow from all the discussion above in this section.)

We first define $c(s, s')$ for $w(s), w(s')$, $s < s'$ in the same interval I_t , by cases (i)-(iv) above:

$$\begin{aligned} \text{Case (i)} \quad & c_0(s, s') = 0, \\ \text{Case (ii)} \quad & c_0(s, s') = z_0(w(s')) - z_0(w(s)), \\ \text{Case (iii)} \quad & c_0(s, s') = \begin{cases} z_{0,t}(w(s')) + l - z_{0,t}(w(s)) & s' > t \geq s \\ 0 & \text{otherwise,} \end{cases} \\ \text{Case (iv)} \quad & c_0(s, s') = 0. \end{aligned}$$

Clearly $c_0(s, s') \in \{-1, 0, 1\}$. Now assume $I_{t_0}, I_{t_1}, \dots, I_{t_j}$ is a finite subcover of $[0, S]$, and $0 = s_0 < s_1 < \dots < s_{j^*} = S$ such that s_{i-1}, s_i are in the same open interval I_{t_i} . We define

$$c_0(0, S) = \sum_{i=0}^{j^*-1} c_0(s_{i-1}, s_i).$$

It is now straightforward to verify that the definition is independent of the choice of t_i , and that c_i, d_i satisfy (A1)-(A4).

11.4. Proof of Lemma 3. We slightly adapt [5], Proposition 2.1. Let C be the uniform bound on $|c_i(t)|$, and let $\bar{w}_i(t) = \exp(Ct)w_i(t)$. Then $\bar{w}_i(t)$ is a solution of an equation of type (3.1) with $\bar{a}_i = a_i, \bar{b}_i = b_i, \bar{c}_i = c_i + C$, thus $\bar{a}_i, \bar{b}_i, \bar{c}_i \geq 0$ satisfying (3.2). Write it shortly

$$(11.8) \quad d\bar{w}/dt = \bar{A}(t)\bar{w}(t),$$

where $\bar{A}(t)$ is an infinite matrix. By standard results on existence of solutions on Banach spaces, the solution of (11.8) exists locally in any of the norms (3.3) and is the limit of Picard iterations

$$\bar{w}^{(n)}(t) = \bar{w}(0) + \int_0^t \bar{A}(s)\bar{w}^{(n-1)}(s)ds,$$

$\bar{w}^{(0)}(t) = \bar{w}(0)$. As $\bar{A}(s)$ has non-negative elements, we easily see that $\bar{w}(0) \geq 0$ implies $\bar{w}(t) \geq 0$ for $t \geq 0$, and $w_i(t)$ and $\bar{w}_i(t)$ have the same sign.

12. APPENDIX 2: ON ZEROES OF DEGREE ∞

This Appendix is dedicated to proofs of Lemmas 6 and 8. Assume without loss of generality that the zero of degree ∞ of $w(0) = u(0) - v(0)$ is at $i = 1$, i.e. $w_0(0) \neq 0$, $w_i(0) = 0$ for all $i \geq 1$. We first consider the case when all u_i are strictly monotone on either sides of $t = 0$. The proof is somewhat combinatorial: we consider different alternatives for the signs of du_j/dt .

Lemma 20. *Assume all as in Lemma 6. Assume also that for some $i_0 \geq 1$, there exist $\delta_j > 0$ so that du_j/dt is not zero on $(-\delta_j, 0)$ and $(0, \delta_j)$ for all $j \geq i_0$. Then the conclusion of Lemma 6 holds.*

Proof. First note that the conclusion of Lemma 19 holds for $w(0) = u(0) - v(0)$, and recall that $u_j(0) = v_j(0)$ for $j \geq 1$.

Case 1. Assume that for some $j_1 \geq i_0$, the signs of $du_{j_1}/dt, \dots, du_{j_1+3}/dt$ are $+, +, +, +$ (analogously we show $-, -, -, -$) for $t \in (-\delta_{j_1}, 0), \dots, (-\delta_{j_1+3}, 0)$. Now because of (11.7), for some $\varepsilon > 0$ small enough, for any $t_0 \in (-\varepsilon, 0)$, $u_{j_1}(t_0) > v_{j_1}(t_0)$, $u_{j_1+1}(t_0) < v_{j_1+1}(t_0)$, $u_{j_1+2}(t_0) > v_{j_1+2}(t_0)$; or $u_{j_1+1}(t_0) > v_{j_1+1}(t_0)$, $u_{j_1+2}(t_0) < v_{j_1+2}(t_0)$, $u_{j_1+3}(t_0) > v_{j_1+3}(t_0)$. Consider without loss of generality the

first possibility and fix some $0 > t_0 > -\varepsilon$. Now find $0 > t_1 > t_0$ so that $u_{j_1+1}(t_1) = v_{j_1+1}(t_0)$. As by assumptions $u_{j_1}(t_1) > v_{j_1}(t_0)$ and $u_{j_1+2}(t_1) > v_{j_1+2}(t_0)$, we see that $u(t_1) - v(t_0)$ has a singular zero of degree 1 at $j_1 + 1$.

Case 2. Assume that for some $i_0 \leq j_1 < j_2 < j_3 < j_4$ (not necessarily consecutive), the signs of $du_{j_1}/dt, \dots, du_{j_4}/dt$ alternate on $(0, \delta_{j_1}), \dots, (0, \delta_{j_4})$, say $-, +, -, +$ (analogously $-, +, -, +$). By (11.7), either $u_j(t) > v_j(t)$ for all $j \geq i_0$ and all $t \in (0, \delta_j)$, or $u_j(t) < v_j(t)$, consider again only the first possibility. Fix $0 < t_0 < \varepsilon$ for ε small enough, and then we can find $0 < t_2 < t_0$ so that $u_{j_2}(t_2) = v_{j_2}(t_0)$. Thus we can find $t_2 \leq t_1 < t_0$, as the smallest such time such that $u_j(t_1) \geq v_j(t_0)$ for all $j = j_1 + 1, \dots, j_3 - 1$, and then for $t = t_1$ at least one of the inequalities is $=$. As by assumptions $u_{j_1}(t_1) > v_{j_1}(t_0)$, $u_{j_3}(t_1) > v_{j_3}(t_0)$, we see that $u(t_1) - v(t_0)$ has a singular zero of finite degree.

Case 3. Assume that for some $j_1 \geq i_0$, $k \geq 2$ the signs of $du_{j_1}/dt, \dots, du_{j_1+k}/dt$ have the signs $-, +, +, \dots, +, -$ on $(-\delta_j, 0)$ and the signs $-, -, \dots, -$ on $(0, \delta_j)$ (or the alternate sign case). Fix $0 < t_0 < \varepsilon$ for ε small enough, and find $-\varepsilon < t_1 < 0$ as the largest $t < 0$ so that $u_j(t) \geq u_j(t_0)$ for all $j = j_1 + 1, \dots, j_1 + k - 1$, and then for ε small enough at least one of inequalities is $=$. As by assumptions, $u_{j_1}(t_1) > u_{j_1}(t_0)$, $u_{j_1+k}(t_1) > u_{j_1+k}(t_0)$, $u(t_1) - u(t_0)$ has a singular zero of a finite degree.

Now it is straightforward to check that one of the cases 1-3 (including the alternate possibilities) must hold. \square

Proof of Lemma 6. We consider separately the DC and AC case. In the DC case, we use the fact that all the results on the zero-set apply to $w = du/dt$. As this does not necessarily hold in the AC case, we apply real analyticity of the solution.

Consider the DC case. If $du_j(0)/dt \neq 0$, then u_j is strongly monotone in a neighborhood of $t = 0$. Also if $du_j(0)/dt = 0$, by Lemmas 17, 18 and 19, $du_j/dt = d_j t^k + o(t^k)$ for some $d_j \neq 0$ and an integer $k \geq 1$ in all cases except possibly when $(j, 0)$ is a part of a zero of Type II of an even degree of $w = du(0)/dt$. If $w = du(0)/dt$ has only finitely many zeroes of Type II, even degree, on $(i, 0)$ for $i \geq 1$, we see from this discussion that for some $i \geq i_0$ the assumptions of Lemma 20 hold and the proof is completed. Now assume the contrary, and consider a zero of Type II, even degree of $w = du(0)/dt$, say at $(j+1, 0), \dots, (j+k, 0)$. Then $du_j(0)/dt$ and $du_{j+k+1}(0)/dt$ have alternate, non-zero signs. As there are infinitely many such zeroes, we take any three and obtain the Case 2 in the proof of Lemma 20.

Consider now the AC case, and assume without loss of generality $F(t)$ in (1.1) is not constant (if it is, this reduces to the DC case). By assumptions and Lemma 4, $u_j(t)$ is real analytic on every open subset of the set of times for which $u(t) \in \mathcal{X}$. Then we have either

$$(12.1) \quad du_j/dt = d_j t^k + o(t^k), \quad k \geq 0$$

or $du_j/dt = 0$ in a neighborhood of $t = 0$ (see e.g. [17], Remark 1.2.13). If for all $j \geq i_0$, the first possibility holds, we can apply Lemma 20 and complete the proof. Assume the contrary, i.e. existence of infinitely many $j \geq 1$ so that du_j/dt locally vanishes around $t = 0$, and discuss all the possibilities. 1) If there are three or more consecutive vanishing du_j/dt , inserting it into the derivative of (1.1) implies that $f(t)$ is locally constant, thus as real analytic and periodic, constant everywhere, which is a contradiction. 2) Assume there are exactly two consecutive vanishing $du_{j+1}/dt, du_{j+2}/dt$, for some $t \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is small enough such that $du_j/dt, du_{j+3}/dt$ have no zeroes on $(0, \varepsilon)$ (this is possible as they satisfy (12.1)). Then for any $0 < t_0 < t_1 < \varepsilon$, clearly $u(t_0) - u(t_1)$ has a zero of degree 2 at $j+1, j+2$. 3) Assume there is a vanishing du_{j+1}/dt on $t \in (-\varepsilon, \varepsilon)$ such that du_j/dt and du_{j+2}/dt have no zeroes for $t \in (0, \varepsilon)$ and have the same sign. Then

analogously for any $0 < t_0 < t_1 < \varepsilon$, $u(t_0) - u(t_1)$ has a singular zero of degree 1 at $j + 1$. 4) Finally, consider infinitely many vanishing du_j/dt so that the signs of du_{j-1}/dt , du_{j+1}/dt are not zero and alternate for $t \in (0, \delta_j)$. But this is the case 2 from the proof of Lemma 20. As we have exhausted all the possibilities, the proof is completed. \square

Proof of Lemma 8. We can apply Lemma 19 to $w = du/dt$ and immediately see that the Case 3 of the proof of Lemma 20 holds. So the conclusion of the case 3 must hold, which completes the proof. \square

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